CHAPTER

11

Matrices

- What is a matrix?
- How is the order of a matrix defined?
- How are the positions of the elements of a matrix specified?
- What are the rules for adding and subtracting matrices?
- How do we multiply a matrix by a scalar?
- What is the method for multiplying a matrix by another matrix?
- What are the properties of the identity and inverse matrices?
- How can your graphics calculator be used to do matrix operations?

11.1 What is a matrix?

A **matrix** (plural **matrices**) is a rectangular array of numbers set out in rows and columns. Matrices can be used to store information, solve sets of simultaneous equations, find optimal solutions in business, analyse networks, transform shapes in geometry, encode information and devise the best strategies in game theory. We will explore some of these applications while learning the basic theory of matrices.

For example, suppose a market stall operates on Friday and Saturday. Sales could be recorded using matrix A.

$$A = \begin{matrix} \\ \textit{Friday} \\ \textit{Saturday} \\ \\ \end{matrix} \begin{matrix} \textit{Shirts} & \textit{Jeans} & \textit{Belts} \\ \left[\begin{matrix} 6 \\ 3 \end{matrix}\right. & \begin{matrix} 8 \\ 7 \end{matrix} & \left.\begin{matrix} 4 \\ 1 \end{matrix}\right] \\ \textit{column 1} & \textit{column 2} & \textit{column 3} \end{matrix} \begin{matrix} \\ \textit{row 1} \\ \textit{row 2} \\ \\ \end{matrix}$$

Friday sales are given in row 1.
Saturday sales are found in row 2.

The number of shirts sold is listed in column 1.
The number of jeans sold is listed in column 2.
The number of belts sold is listed in column 3.

We can read the following information from the matrix:

- On Friday 8 jeans were sold.
- On Saturday 1 belt was sold.
- The total number of items sold on Friday was $6 + 8 + 4 = 18$.
- The total number of belts sold was $4 + 1 = 5$.

Order of a matrix

The **order** (or shape) of a matrix is written as:

number of **rows** $\times$ number of **columns**

row 1 $\begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix}$ row 2

Think: ' rows in a cinema'

$\begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix}$

column 1 column 2 column 3

Think: 'columns of the Parthenon'

The order of matrix A in the market stall example above is 2×3; that is, 2 rows $\times$ 3 columns. The number of rows is always given first, then the number of columns. Rows first, then columns.

Remember: When you walk into a cinema you go to your *row first*.

Matrices are usually labelled using capital letters, such as A, B, O, etc.

Elements of a matrix

The numbers within a matrix are called its **elements**.

Position of an element in a matrix

$a_{i,j}$ is the **element** in row i, column j.

In matrix A of our example, the element $a_{2,1}$ is in row 2, column 1. See that $a_{2,1} = 3$, and it tells us that on Saturday 3 shirts were sold.

Example 1 Identifying the elements of a matrix

Matrix B shows the numbers of boys and girls in Years 10 to 12 at a particular school.

a Give the order of matrix B.

b What information is given by the element $b_{1,2}$?

c Which element gives the number of girls in Year 12?

$B =$

	Boys	Girls
Year 10	57	63
Year 11	48	51
Year 12	39	45

Solution

a Count the rows, count the columns.
Remember: Order is rows $\times$ columns.

The order of matrix B is 3×2.

b The element $b_{1,2}$ is in *row 1* and *column 2*. This is where the *Year 10 row* meets the *Girls column*.

There are 63 girls in Year 10.

c *Year 12* is *row 3*. *Girls* are *column 2*.

The number of Year 12 girls is given by $b_{3,2}$.

How to enter a matrix into your graphics calculator

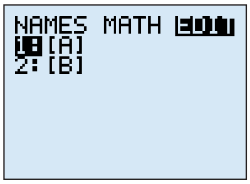
Enter the matrix B into a graphics calculator.

$$B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$$

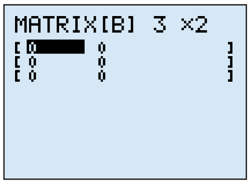
Show how your graphics calculator can identify the element $b_{2,1}$.

Steps

1 Press [2nd] [MATRIX], the second function of the [x^{-1}] key.
Then press [▶] [▶] to display the **MATRIX EDIT** menu.

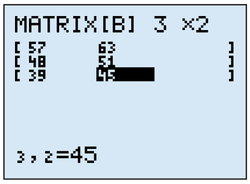


2 Press [2] to select item 2, the matrix B.



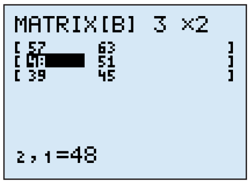
3 Press [3] [ENTER] [2] [ENTER] to set up a 3 × 2 matrix.

4 Type 57 and press [ENTER] to enter **57** as element $b_{1,1}$.



5 Type the remaining values, 63, 48, 51, 39, 45, pressing [ENTER] after each value so that it is entered and you move to the next position.
Entries are made working from left to right, then starting the next row below.
As each element is highlighted, its row and column are shown at the bottom of the screen.

6 Press [▲] [◀] to move to row 2, column 1.
See that $b_{2,1} = 48$.



7 Press [2nd] [QUIT], the second function of the [MODE] key, to return to the home screen.

8 Write your answer.

The element $b_{2,1} = 48$.

Exercise 11A

1 Matrix C is shown.

$$C = \begin{bmatrix} 2 & 4 & 16 & 7 \\ 6 & 8 & 9 & 3 \\ 5 & 6 & 10 & 1 \end{bmatrix}$$

a Give the order of the matrix.

b State the value of: **i** $c_{1,3}$ **ii** $c_{2,4}$ **iii** $c_{3,1}$.

2 For each of the following matrices:

i state the order **ii** find the values of the required elements.

a $A = \begin{bmatrix} 5 & 6 & 8 \\ 4 & 7 & 9 \end{bmatrix}$ Find $a_{1,2}$ and $a_{2,2}$. **b** $B = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix}$ Find $b_{1,3}$ and $b_{1,1}$.

c $C = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 8 & -4 \end{bmatrix}$ Find $c_{3,2}$ and $c_{1,2}$. **d** $D = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$ Find $d_{3,1}$ and $d_{1,1}$.

e $E = \begin{bmatrix} 10 & 12 \\ 15 & 13 \end{bmatrix}$ Find $e_{2,1}$ and $e_{1,2}$. **f** $F = \begin{bmatrix} 8 & 11 & 6 & 2 \\ 4 & 1 & 5 & 7 \\ 6 & 14 & 17 & 20 \end{bmatrix}$ Find $f_{3,4}$ and $f_{2,3}$.

3 For matrix D, give the values of the following elements.

a $d_{2,3}$ **b** $d_{4,5}$ **c** $d_{1,1}$ **d** $d_{2,4}$ **e** $d_{4,2}$

$$D = \begin{bmatrix} 3 & 4 & 6 & 11 & 2 \\ 5 & 1 & 9 & 10 & 4 \\ 8 & 7 & 2 & 0 & 1 \\ 6 & 8 & 5 & 8 & 2 \end{bmatrix}$$

4 Enter the matrix in Question **3** into your graphics calculator. Use it to check your answers to Question **3**.

5 Some students were asked which of four sports they preferred to play and the results were entered in the following matrix.

a How many Year 11 students preferred basketball?

b Give the order of matrix S.

c What information is given by $s_{2,3}$?

$$S = \begin{array}{r} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tennis} & \textit{Basketball} & \textit{Football} & \textit{Hockey} \end{array} \\ \begin{bmatrix} 19 & 18 & 31 & 14 \\ 16 & 32 & 22 & 12 \\ 21 & 25 & 5 & 7 \end{bmatrix} \end{array}$$

6 For the following matrices:

a give the order of each matrix.

b identify the elements: $a_{3,2}$, $b_{2,1}$, $c_{1,1}$ and $d_{2,4}$ of matrices A, B, C and D respectively.

$$A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 5 & 3 \\ -3 & 4 & 8 \\ 7 & 6 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ -5 \end{bmatrix} \quad C = \begin{bmatrix} 8 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -3 & 0 & 1 & 9 \\ 6 & 11 & 2 & 7 & 5 \end{bmatrix}$$

7 Matrix F shows the number of hectares of land used for different purposes on two farms, X and Y.

Row 1 represents Farm X and *row 2* represents Farm Y. *Columns 1*, *2* and *3* show the amount of land used for wheat, cattle and sheep (W, C, S) respectively, in hectares.

$$F = \begin{array}{c} \begin{array}{ccc} W & C & S \end{array} \\ \begin{bmatrix} 150 & 300 & 75 \\ 200 & 0 & 350 \end{bmatrix} \end{array} \begin{array}{c} \\ X \\ Y \end{array}$$

a How many hectares are used on:

i Farm X for sheep?

ii Farm X for cattle?

iii Farm Y for wheat?

b Calculate the total number of hectares used on both farms for wheat.

c State the information that is given by:

i $f_{2,2}$ **ii** $f_{1,3}$ **iii** $f_{1,1}$.

d Which element of matrix F gives the number of hectares used:

i on Farm Y for sheep?

ii on Farm X for cattle?

iii on Farm Y for wheat?

e State the order of matrix F.

8 An agency has an allocation of tickets for sale for a concert to be held on two different nights. For Saturday it has 100 \$70 tickets, 150 \$80 tickets and 175 \$90 tickets. For Sunday it has 60 at \$70, none at \$80 and 200 at \$90.

a Display this information in a 2×3 matrix.

b Calculate the total value of the tickets the agency has for sale on:

i Saturday **ii** Sunday **iii** both nights.

11.2 Networks

A **network** is a diagram of points (vertices) joined by lines (edges). It can be used to show connections or relationships.

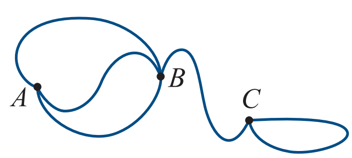
A matrix can be used to represent a network and solve related problems.

Example 2 Using a matrix to represent a network

The network drawn shows the ways to travel between three towns, A, B and C.

a Use a matrix to represent the network. Each element should tell the number of ways to travel directly from one town to another.

b Use the matrix to find which town has the most ways of travelling to or from it.



Solution

a

1 In the first row:

From A to A there are 0 ways.

From A to B there are 3 ways.

From A to C there are 0 direct ways.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} 0 & 3 & 0 \\ & & \\ & & \end{array}\right] \end{array}$$

2 In the second row:
From B to A there are 3 ways.
From B to B there are 0 ways.
From B to C there is 1 way.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ & & \end{bmatrix} \end{array}$$

3 In the third row:
From C to A there are 0 direct ways.
From C to B there is 1 way.
From C to C there are 2 ways. You can leave by the upper exit and return by the lower entrance, or vice versa.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

b

1 The ways of leaving a town are shown by the sum of the elements in its row.
B has the greatest row total. There are 4 ways of leaving B.

Row B sum $= 3 + 0 + 1 = 4$

2 The ways of arriving at a town are shown by the sum of the elements in its column.
B has the greatest column total. There are 4 ways of arriving at B

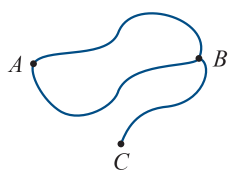
Column B sum $= 3 + 0 + 1 = 4$

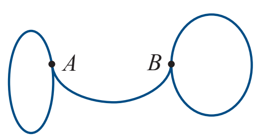
3 Write your answer.

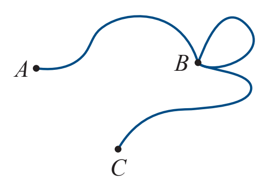
Town B has the most ways of travelling to or from it.

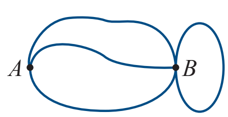
Exercise 11B

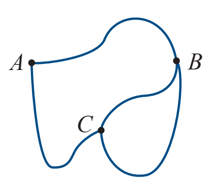
1 Write the matrix to represent each network.

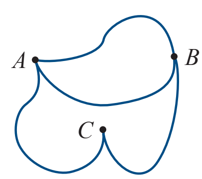
a


b


c


d


e


f


2 Draw a network for each matrix.

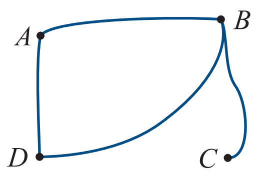
a $\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$

b $\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$

c $\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \end{array}$

d $\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} \end{array}$ **e** $\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \end{array}$ **f** $\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix} \end{array}$

3 The network opposite has lines showing which people from the four people *A*, *B*, *C* and *D* have met.



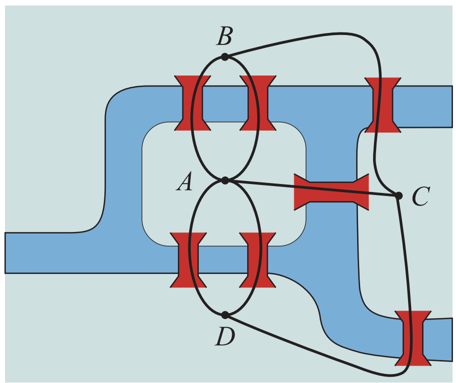
a Represent the network using a matrix. Use 0 when two people have *not* met and 1 when they have met.

b How can the matrix be used to tell who has met the most people?

c Who has met the most people?

d Who has met the least number of people?

4 You met the following problem in Chapter 10, 'Networks'. In the city of Königsberg, seven bridges connected the sides of the river to two islands. The citizens wondered whether it was possible to plan a walk in which every bridge was crossed just once. The mathematician Euler analysed the problem by first representing it using a network diagram. Each side of the river and each island was represented by a point (vertex) *A*, *B*, *C* or *D*. The connecting bridges were shown as lines (edges).



a Use a matrix to represent Euler's network.

b How can the matrix be used to tell which land-mass had the most connections to it? Refer to your work in Chapter 10.

c How do the row and column totals of the matrix relate to the degrees of the vertices of the network?

d Using just the matrix, and your knowledge of what makes a network traversable, could you solve the Königsberg Bridges problem? Explain your answer.

11.3 Equal matrices

Two matrices are **equal** if they have the same order and their corresponding elements are equal. In other words, they must have the *same numbers* in the *same positions*.

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \quad \text{But} \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \neq \begin{bmatrix} 6 & 5 \\ 8 & 7 \end{bmatrix}$$

Example 3 **Equal matrices**

The two matrices shown are equal. Find the values of x, y, w and z.

$$\begin{bmatrix} w & 2x \\ (y+1) & 8 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 12 & z \end{bmatrix}$$

Strategy: The matrices are equal so elements in the same position are equal. Equate the elements that are in the same positions, then solve any equations.

Solution

1 w is in the same position as 4. $2x$ is in the same position as 6. $y+1$ is in the same position as 12. 8 is in the same position as z.

$w = 4$ $\quad 2x = 6$

$x = 3$

2 Solve the equations for x and y, and reverse the equation for z.

$y + 1 = 12$ $\quad 8 = z$

$y = 11$ $\quad z = 8$

3 Write your answers.

So $x = 3$, $y = 11$, $w = 4$, $z = 8$.

Exercise 11C

1 The following matrices are equal. Find the values of a, b, c and d.

a $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ -2 & 6 \end{bmatrix}$

b $\begin{bmatrix} a & 3b \\ (c+2) & -1 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 7 & d \end{bmatrix}$

c $[a \quad (b-1) \quad 4c \quad 3] = [2 \quad 9 \quad 20 \quad d]$

d $\begin{bmatrix} 2a \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ (b+1) \end{bmatrix}$

2 Find the values of a, b, c, d and e. $\begin{bmatrix} a & 6 & 8 \\ -c & -4 & -9 \end{bmatrix} = \begin{bmatrix} 5 & 6 & b \\ 3 & 2d & -e \end{bmatrix}$

11.4 Adding and subtracting matrices

Matrices are added by adding the elements that are in the same positions. Subtraction is done by subtracting the elements that are in the same positions. This can only be done if the two matrices have the **same order**.

Example 4 **Addition and subtraction of matrices**

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$

b $\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$

Solution

a

1 Write the addition.

$$\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$$

2 Add the elements that are in the same positions.

$$= \begin{bmatrix} 2+9 & 4+8 \\ 5+9 & 1+(-1) \end{bmatrix}$$

3 Evaluate each element.

$$= \begin{bmatrix} 11 & 12 \\ 14 & 0 \end{bmatrix}$$

b

1 Write the subtraction.

$$\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$$

2 Subtract the elements that are in the same positions.

$$= \begin{bmatrix} 7-4 & 3-2 \\ 2-(-1) & 8-9 \\ 1-3 & 0-7 \end{bmatrix}$$

3 Evaluate each element.

$$= \begin{bmatrix} 3 & 1 \\ 3 & -1 \\ -2 & -7 \end{bmatrix}$$

The zero matrix, *O*

Any matrix with zero as the value of each element is called a **zero matrix**. The following are examples of zero matrices.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Just as in arithmetic with ordinary numbers, adding or subtracting a zero matrix does not make any change to the original matrix. For example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Also, subtracting any matrix from itself gives a zero matrix. For example:

$$\begin{bmatrix} 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Exercise 11D

1 **a** $\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}$

c $\begin{bmatrix} 3 & 5 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 9 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e $\begin{bmatrix} 8 & 6 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & 4 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 2 & -8 \end{bmatrix}$

g $\begin{bmatrix} 4 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 5 \end{bmatrix}$ **h** $\begin{bmatrix} 7 & -5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \end{bmatrix}$

i $\begin{bmatrix} 4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 3 \end{bmatrix}$ **j** $\begin{bmatrix} 4 & -3 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -5 & -1 & 8 \end{bmatrix}$

2 Using the matrices given:

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 2 \\ 1 & 0 \\ 3 & -8 \end{bmatrix} \quad D = \begin{bmatrix} -3 & 5 \\ 4 & -2 \\ 1 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

find, where possible:

a $A + B$ **b** $B + A$ **c** $A - B$ **d** $B - A$
e $B + E$ **f** $C + D$ **g** $B + C$ **h** $D - C$

3 Two people shared the work of a telephone poll surveying voting intentions. The results for each person's survey are given in matrix form.

Sample 1:

	Liberal	*Labor*	*Democrat*	*Green*
Men	19	21	7	3
Women	18	17	11	4

Sample 2:

	Liberal	*Labor*	*Democrat*	*Green*
Men	24	21	3	2
Women	19	20	6	5

Write a matrix showing the overall result of the survey.

4 The weights and heights of four people were recorded, then checked again 1 year later.

2004 results:

	Aida	*Bianca*	*Chloe*	*Donna*
Weight (kg)	32	44	59	56
Height (cm)	145	155	160	164

2005 results:

	Aida	*Bianca*	*Chloe*	*Donna*
Weight (kg)	38	52	57	63
Height (cm)	150	163	167	170

a Write the matrix that gives the changes in each person's weight and height after 1 year.
b Who gained the most weight?
c Which person had the greatest height increase?

11.5 Scalar multiplication

When a matrix is multiplied by a number, each element in the matrix is multiplied by that number. This is called **scalar multiplication**.

Example 5 Scalar multiplication

Use scalar multiplication to simplify this matrix. $3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$

Solution

1 Multiply each number in the matrix by 3.

$$3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 3\times 5 & 3\times 1 \\ 3\times -3 & 3\times 0 \end{bmatrix}$$

2 Evaluate each element.

$$= \begin{bmatrix} 15 & 3 \\ -9 & 0 \end{bmatrix}$$

Scalar multiplication has practical applications.

Example 6 Application of scalar multiplication

A gymnasium has the enrolments in courses shown in this matrix.

$$\begin{array}{c} \\ \textit{Men} \\ \textit{Women} \end{array} \begin{array}{ccc} \textit{Body building} & \textit{Aerobics} & \textit{Fitness} \\ \end{array}$$
$$\begin{array}{l} \textit{Men} \\ \textit{Women} \end{array}\begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix}$$

The manager wishes to double the enrolments in each course. Show this in a matrix.

Solution

1 Each element is multiplied by 2.

$$2\times\begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix} = \begin{bmatrix} 2\times 70 & 2\times 20 & 2\times 80 \\ 2\times 10 & 2\times 50 & 2\times 60 \end{bmatrix}$$

2 Evaluate each element.

$$= \begin{array}{l} \text{Men} \\ \text{Women} \end{array}\overset{\begin{array}{ccc}\text{Body building} & \text{Aerobics} & \text{Fitness}\end{array}}{\begin{bmatrix} 140 & 40 & 160 \\ 20 & 100 & 120 \end{bmatrix}}$$

Scalar multiplication can be used in conjunction with addition and subtraction of matrices.

Example 7 Scalar multiplication and subtraction of matrices

If $A = \begin{bmatrix} 5 & 1 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$, find the matrix equal to $2A - 3B$.

Solution

1 Write $2A - 3B$ in expanded matrix form.

$$2A - 3B = 2 \times \begin{bmatrix} 5 & 1 \\ 3 & -4 \end{bmatrix} - 3 \times \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$$

2 Multiply the elements in A by 2 and the elements in B by 3.

$$= \begin{bmatrix} 10 & 2 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 18 & -15 \end{bmatrix}$$

3 Subtract the elements in corresponding positions.

$$= \begin{bmatrix} 4 & 5 \\ -12 & 7 \end{bmatrix}$$

How to do addition, subtraction and scalar multiplication of matrices using a graphics calculator

Given the matrices: $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$

use a graphics calculator to find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

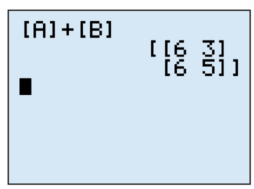
a $A + B$

1 Enter matrix A and matrix B into your graphics calculator.

If you need help, see the section. 'How to enter a matrix into your graphics calculator' (page 422).

2 Press [2nd] [MATRIX] [1] to select matrix A. Press [+], then [2nd] [MATRIX] [2] for matrix B.

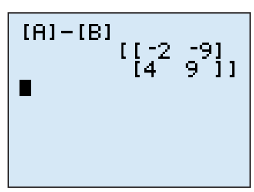
3 Press [ENTER]. Ignore the extra square brackets.



b $A - B$

1 Press [2nd] [ENTRY], the second function of the [ENTER] key, to recall the previous entry **[A] + [B]**.

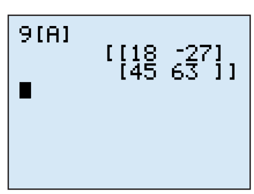
2 Move the flashing cursor over the **+** sign, then press [−] [ENTER] to change it to minus.



c $9A$

1 Type 9, then press [2nd] [MATRIX] [1] for matrix A.

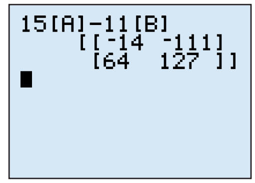
2 Press [ENTER].



d $15A - 11B$

1 Type 15, then press [2nd] [MATRIX] [1] for matrix A.
Type 11, then press [2nd] [MATRIX] [2] for matrix B.

2 Type [ENTER]



Exercise 11E

1 Calculate the values of the following.

a $2\begin{bmatrix} 7 & -1 \\ 4 & 9 \end{bmatrix}$ **b** $5\begin{bmatrix} 0 & -2 \\ 5 & 7 \end{bmatrix}$ **c** $-4\begin{bmatrix} 16 & -3 \\ 1.5 & 3.5 \end{bmatrix}$ **d** $1.5\begin{bmatrix} 1.5 & 0 \\ -2 & 5 \end{bmatrix}$

e $3\begin{bmatrix} 6 & 7 \end{bmatrix}$ **f** $6\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **g** $\frac{1}{2}\begin{bmatrix} 4 & 6 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ **h** $-1\begin{bmatrix} 3 & 6 & -8 \end{bmatrix}$

2 Given the matrices:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 6 \\ 1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 4 \\ -2 & -5 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

find the value of:

a $3A$ **b** $2B + 4C$ **c** $5A - 2B$ **d** $2O$ **e** $3B + O$

3 Enter the matrices A and B into your graphics calculator.
Find the matrices equal to:

$$A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$$

a $17A - 14B$ **b** $29B - 21A$ **c** $9A + 7B$ **d** $3(5A - 4B)$

4 The expenses arising from costs and wages for each section of three stores, A, B and C, are shown in the Costs matrix. The Sales matrix shows the money from the sale of goods in each section of the three stores. Figures represent the nearest million dollars.

Costs:

	Clothing	*Furniture*	*Electronics*
Store A	12	10	15
Store B	11	8	17
Store C	15	14	7

Sales:

	Clothing	*Furniture*	*Electronics*
Store A	18	12	24
Store B	16	9	26
Store C	19	13	12

a Write a matrix showing the profits in each section of each store.

b If 30% tax must be paid on profits, show the amount of tax that must be paid by each section of each store. No tax needs to be paid for a section that has made a loss.

11.6 Matrix multiplication

Matrix multiplication is the multiplication of a matrix by another matrix. We will see that in some cases a matrix can be multiplied by itself. That would be like squaring a number in arithmetic. Be careful not to confuse matrix multiplication with the scalar multiplication done in the previous section. Scalar multiplication is the multiplication of a matrix by a *number*, so that every element in the matrix is multiplied by that number.

The matrix multiplication of two matrices A and B can be written as $\boldsymbol{A} \times \boldsymbol{B}$ or just $\boldsymbol{AB}$. Although it is called multiplication and the symbol $\times$ may be used, matrix multiplication is not the simple multiplication of numbers but a routine involving the sum of pairs of numbers that have been multiplied.

For example, the method of **matrix multiplication** can be demonstrated by using a practical example. The numbers of CDs and DVDs sold by Fatima and Gaia are recorded in matrix N. The selling prices of the CDs and DVDs are shown in matrix P.

$$N = \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{c} \textit{CDs}\ \ \textit{DVDs} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{l} \\ \textit{CDs} \\ \textit{DVDs} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 20 \\ 30 \end{bmatrix} \end{array}$$

We want to make a matrix, S, that shows the value of the sales made by each person.

Fatima sold: 7 CDs at \$20 + 4 DVDs at \$30.
Gaia sold: 5 CDs at \$20 + 6 DVDs at \$30.

$$S = \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 4 \times 30 \end{bmatrix} \end{array}$$

The commonsense steps used in this example follow the routine for the matrix multiplication of $N \times P$.

As we move across the *first row* of matrix N we move down the *column* of matrix P, adding the products of the pairs of numbers as we go.

Then we move across the *second row* of matrix N and down the *column* of matrix P, adding the products of the pairs of numbers as we go.

$$\begin{array}{c} N \times P \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} \end{array} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ \cdots \quad \cdots \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix}$$

$$= \begin{bmatrix} 140 + 120 \\ 100 + 180 \end{bmatrix}$$

$$= \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 260 \\ 280 \end{bmatrix} \end{array}$$

Rules for matrix multiplication

Because of the way the products are formed, **the number of columns in the first matrix must equal the number of rows in the second matrix**. Otherwise we say that matrix multiplication is **not defined**, meaning it is not possible.

For matrix multiplication to be defined:

order of 1st matrix $m \times n$ order of 2nd matrix $n \times p$

must be the same

Think of the orders as two railway carriages that must be the same where they meet.

In our example of the CD and DVD sales:

order of 1st matrix 2×2 order of 2nd matrix 2×1

the same

Notice that the outside numbers give the order of the **product matrix**: the matrix made by multiplying the two matrices. In our case, the answer is a 2×1 matrix.

The **order of the product matrix** is given by:

order of 1st matrix $m \times n$ order of 2nd matrix $n \times p$

order of answer $m \times p$

Think: when the 'railway carriages' meet, the result has an order given by the end numbers.

We will check that these two important rules hold in the examples that follow.

For example, back at Fatima and Gaia's music shop there is now a special sales promotion of one cinema ticket with each CD purchased and two cinema tickets with each DVD. Matrix P can be modified to include this information. We will use the previous figures for the numbers of CDs and DVDs sold, so that it is easier to see the pattern made by the new information about free cinema tickets.

$$N = \begin{matrix} & \begin{matrix} CDs & DVDs \end{matrix} \\ \begin{matrix} Fatima \\ Gaia \end{matrix} & \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{matrix} \qquad P = \begin{matrix} & \begin{matrix} \$ & Tickets \end{matrix} \\ \begin{matrix} CDs \\ DVDs \end{matrix} & \begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} \end{matrix}$$

Matrix multiplication can be used to make a matrix that displays the value of the sales and the number of tickets issued by each person.

As we move across the *first row* of matrix N we move down the *first column* of matrix P, adding the products of the pairs of numbers as we go.

$$N \times P$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Now move across the *first row* of matrix N again, but this time go down the *second column* of matrix P, adding the products of the pairs of numbers.

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ & \end{bmatrix}$$

There are no more columns in matrix P to go down, so start the process again by going across the *second row* of the matrix N and down the *first column* of matrix P, adding the products of the pairs of numbers.

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ 5 \times 20 + 6 \times 30 & \cdots \end{bmatrix}$$

Now move across the *second row* of matrix N again and go down the *second column* of matrix P, adding the products of the pairs of numbers.

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ 5 \times 20 + 6 \times 30 & 5 \times 1 + 6 \times 2 \end{bmatrix}$$

Tidy up by doing a bit of arithmetic.

$$= \begin{bmatrix} 140 + 120 & 7 + 8 \\ 100 + 180 & 5 + 12 \end{bmatrix}$$

$$= \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \textit{Tickets} \end{array} \\ \begin{bmatrix} 260 & 15 \\ 280 & 17 \end{bmatrix} \end{array}$$

Matrix multiplication of $N \times P$ has given a matrix displaying the value of the sales and the number of tickets issued by each person.

Methods of matrix multiplication

Some people like to think of the matrix multiplication of $A \times B$ using a **run and dive** description.

Matrix multiplication of $A \times B$

The **run and dive** description of matrix multiplication:

Add the products of the pairs made as you:

1. **run** along the first row of A and **dive** down the first column of B
2. repeat running along the first row of A and diving down the next column of B until all columns of B have been used
3. now start running along the next row of A and repeat diving down each column of B
4. repeat this routine until all rows of A have been used.

This procedure can be very tedious and error prone, so we will only do simple cases by hand so that you understand the process. Then a graphics calculator will be used to do matrix multiplication.

Example 8 Matrix multiplication

For the following matrices:

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$$

- decide whether the matrix multiplication in each question below is defined
- if matrix multiplication is defined, give the order of the answer matrix, then do the matrix multiplication.

a AB **b** BA **c** CD

Solution

a AB

1 Write the order of each matrix.

$$\underset{3 \times 2}{A} \qquad \underset{2 \times 1}{B}$$

2 Are the inside numbers the same? Yes.

Matrix multiplication is defined for $A \times B$.

3 The outside numbers give the order of $A \times B$.

The order of $A \times B$ is 3×1.

4 Move across the *first row* of A and down the *column* of B, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ \\ \\ \end{bmatrix}$$

5 Move across the *second row* of A and down the *column* of B, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \end{bmatrix}$$

6 Move across the *third row* of A and down the *column* of B, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ 1 \times 8 + 3 \times 9 \end{bmatrix}$$

7 Tidy up by doing some arithmetic.

$$= \begin{bmatrix} 40 + 18 \\ 32 + 54 \\ 8 + 27 \end{bmatrix}$$

8 Write your answer.

$$\text{So } A \times B = \begin{bmatrix} 58 \\ 86 \\ 35 \end{bmatrix}$$

b BA

1 Write the order of each matrix.

$$\underset{2 \times 1}{B} \qquad \underset{3 \times 2}{A}$$

2 Are the inside numbers the same? No.

Matrix multiplication is not defined for $B \times A$.

c *CD*

1 Write the order of each matrix. — C: 1×3, D: 3×1

2 Are the inside numbers the same? Yes. — Matrix multiplication is defined for $C \times D$.

3 The outside numbers give the order of $C \times D$. — The order of $C \times D$ is 1×1.

4 Move across the *row* of *C* and down the *column* of *D*, adding the products of the pairs.

$$\begin{bmatrix} 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 8 + 4 \times 6 + 7 \times 5 \end{bmatrix}$$

5 Tidy up by doing some arithmetic.

$$= \begin{bmatrix} 16 + 24 + 35 \end{bmatrix}$$

6 Write your answer. — So $C \times D = [75]$

In the previous example, $AB \neq BA$. Usually, when we reverse (**commute**) the order of the matrices in matrix multiplication, we get a different answer. This differs from ordinary arithmetic, where multiplication gives the same answer when the terms are commuted. For example, $3 \times 4 = 4 \times 3$.

> In general, matrix multiplication is **not commutative**. That is,
>
> $$AB \neq BA$$

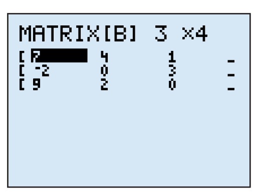
How to multiply matrices using a graphics calculator

Use a graphics calculator to find $A \times B$.

$$A = \begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$$

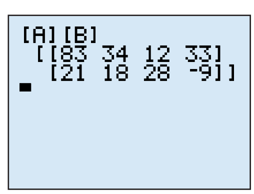
Steps

1 Enter matrix *A* and matrix *B* into your graphics calculator.
(If you need help, see the section: 'How to enter a matrix into your graphics calculator' (page 422).)
Scroll using ▸ or ◂ to see all of matrix *B*.
Press [2nd] [QUIT] to return to the home screen.



2 Press [2nd] [MATRIX] [1] to select matrix *A*.
Press [2nd] [MATRIX] [2] for matrix *B*.
Note: There is no need to put in the multiplication sign as it is assumed.

3 Press [ENTER].



4 Check: A is 2×3 and B is 3×4. So matrix AB should be 2×4.

The product AB is the 2×4 matrix:

5 Write your answer.

$$\begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$$

Only a **square matrix** (number of rows = number of columns) can be multiplied by itself. If A is a square matrix, then press 2nd [MATRIX] 1 x^2 ENTER to find $A \times A$. For higher positive powers of A, use the ^ key.

Exercise 11F

1 For the following matrices:

$$A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 \\ 0 & 8 \\ 2 & -5 \end{bmatrix}$$

- decide whether the matrix multiplication in each question below is defined
- if matrix multiplication is defined, give the order of the answer matrix, then do the matrix multiplication.

a AB **b** BA **c** CB **d** BC
e AA **f** BB **g** AC **h** CA

2 Do these matrix multiplications without using a graphics calculator.

a $\begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 4 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$

d $\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$ **e** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 6 \\ 7 & 8 \end{bmatrix}$

g $\begin{bmatrix} 2 & 5 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ **h** $\begin{bmatrix} 7 & 4 \\ 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **i** $\begin{bmatrix} 5 & 8 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

j $\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$ **k** $\begin{bmatrix} 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$ **l** $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

m $\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **n** $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$ **o** $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

3 Use your graphics calculator to do the matrix multiplications in Question **2**.

4 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ **a** Find AB. **b** Find BA. **c** Does $AB = BA$?

5 Use these matrices to find the required products.

$$C = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

a CD **b** CE **c** CF **d** DE **e** DF

6 Perform the following matrix multiplications using your graphics calculator.

a $\begin{bmatrix} 6 & 8 & 12 \\ 14 & 17 & 11 \end{bmatrix} \begin{bmatrix} 26 & 9 & 21 & 6 \\ 8 & -7 & -4 & 9 \\ 13 & 10 & 5 & 26 \end{bmatrix}$ **b** $\begin{bmatrix} 15 & 9 & 23 & 42 \end{bmatrix} \begin{bmatrix} -6 \\ 22 \\ -8 \\ 19 \end{bmatrix}$

c $\begin{bmatrix} 16 \\ 10 \\ 24 \\ -18 \end{bmatrix} \begin{bmatrix} -31 & 47 & 61 & -14 \end{bmatrix}$ **d** $\begin{bmatrix} 8 & -7 & 9 \\ 6 & 11 & 14 \\ 3 & 21 & -5 \end{bmatrix} \begin{bmatrix} 8 & -19 & 24 \\ 33 & 16 & 19 \\ 4 & 0 & 13 \end{bmatrix}$

e $\begin{bmatrix} 15 & 8 & -4 \\ 21 & -9 & 0 \\ 7 & 11 & 16 \\ -6 & 17 & 27 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 4 & 9 \\ 5 & -7 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

7 Two shop assistants, Ann and Bill, made sales of jeans and shirts as listed in the first matrix. The retail prices of jeans and shirts are given in the second matrix.

$$\begin{array}{c} \\ \textit{Ann} \\ \textit{Bill} \end{array} \begin{array}{c} \textit{Jeans} \quad \textit{Shirts} \\ \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \textit{Jeans} \\ \textit{Shirts} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 60 \\ 45 \end{bmatrix} \end{array}$$

a Obtain a matrix giving the total retail value of the sales of each person.

As a sales promotion, two free hamburger coupons were given with each pair of jeans sold. One coupon was given with each shirt.

b Set up a matrix displaying the prices and coupons given for jeans and shirts.

c Hence derive a matrix showing the value of sales and the number of coupons given by each person.

8 One matrix below shows the number of milkshakes and sandwiches that Helen had for lunch. The number of kilojoules (kJ) present in each food is given in the other matrix.

$$\textit{Helen} \begin{array}{c} \textit{Milkshakes} \quad \textit{Sandwiches} \\ \begin{bmatrix} 2 & 3 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \textit{Milkshakes} \\ \textit{Sandwiches} \end{array} \begin{array}{c} kJ \\ \begin{bmatrix} 1400 \\ 1000 \end{bmatrix} \end{array}$$

Calculate how many kilojoules Helen had for lunch.

9 The first matrix gives the hours for which Tom and Louise agreed to chop firewood and mow the lawns. The second matrix gives the hourly rate of pay for each job.

$$\begin{array}{c} \\ \textit{Tom} \\ \textit{Louise} \end{array} \begin{array}{c} \textit{Chop} \quad \textit{Mow} \\ \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \textit{Chop} \\ \textit{Mow} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 8 \\ 6 \end{bmatrix} \end{array}$$

a How much money will Tom earn by chopping?

b What is the amount Louise will earn by mowing the lawns?

c Write a matrix giving the total earnings of each person.

10

$$\begin{array}{c} \\ \textit{Smith} \\ \textit{Jones} \end{array} \begin{array}{c} \textit{Cars} \quad \textit{Bicycles} \\ \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \textit{Car} \\ \textit{Bicycle} \end{array} \begin{array}{c} \textit{Wheels} \quad \textit{Seats} \\ \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} \end{array}$$

The first matrix above shows the number of cars and bicycles owned by two families. The second matrix records the wheels and seats for cars and bicycles. Find a matrix that gives the numbers of wheels and seats owned by each family.

11 Two bookshops had a special cookbooks promotion. Aubrey's Bookshop sold 35 copies of *The Raw Food Cookbook* and 18 of *Fast Food*. Bettie's Bookshop sold 42 copies of *The Raw Food Cookbook* and 23 of *Fast Food*.

a Set up a matrix showing the sales by each bookshop of the two types of book. Use the rows for the names of the bookshops, and the columns for the names of the books.
The Raw Food Cookbook sold for \$30 and came with two free packets of indigestion tablets. *Fast Food* sold for \$20 and came with three free packets of indigestion tablets.

b Set up a matrix showing the cost and the number of free packets of indigestion tablets for each book. Use the rows for the names of the books, and the columns for the prices and the numbers of free packets.

c Use matrix methods to determine the value of sales and the number of packets of indigestions tablets given out by each shop.

11.7 Identity and inverse matrices

Identity matrix

In ordinary arithmetic the number 1 is called the *multiplicative identity element*. When 1 multiplies a number, the answer is always *identical* to the original number. Is there a matrix that can multiply any matrix and give an answer identical to the original matrix? Consider the following example.

Example 9 **The identity matrix**

$A = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Find AI.
b Find IA.

Solution

a *AI*

1 Write the order of each matrix.
The inside numbers are the same, so matrix multiplication is defined.
The outside numbers tell us that the answer is a 2 × 2 matrix.

$$\begin{matrix} A & I \\ 2\times 2 & 2\times 2 \end{matrix}$$

2 Do the matrix multiplication by hand or using your calculator.

$$A \times I = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 1 + 2\times 0 & 5\times 0 + 2\times 1 \\ 8\times 1 + 3\times 0 & 8\times 0 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

b *IA*

1 Write the order of each matrix.
The inside numbers are the same, so matrix multiplication is defined.
The outside numbers tell us that the answer is a 2 × 2 matrix.

$$\begin{matrix} I & A \\ 2\times 2 & 2\times 2 \end{matrix}$$

2 Do the matrix multiplication by hand or using your calculator.

$$I \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 5 + 0\times 8 & 1\times 2 + 0\times 3 \\ 0\times 5 + 1\times 8 & 0\times 2 + 1\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

Identity matrix for 2 × 2 matrices

The identity matrix for any 2 × 2 matrix *A* is $\mathbf{I} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

where $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$

The identity matrix, *I*, also has the special property that it is **commutative** in matrix multiplication. When *I* is one of the matrices in the multiplication, the answer is the same when the order of the matrices is commuted (reversed).

In Example 10,

$$AI = IA = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

Remember that matrix multiplication is not usually commutative.

Only square matrices have identity matrices. The **identity matrix for any square matrix** is a square matrix of the same order with 1s along the leading diagonal (from the top left to the bottom right) and 0s in all the other positions.

$$[1] \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse matrix

Written as $\boldsymbol{A^{-1}}$, the **inverse** of matrix A is a matrix that multiplies A to make the identity matrix I.

$$\boldsymbol{A \times A^{-1} = I = A^{-1} \times A}$$

For 2×2 matrices, the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The following example finds the inverse of a given matrix.

Finding a rule for the inverse of a matrix

To find the inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider matrix $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

formed by swapping $\boldsymbol{a}$ and $\boldsymbol{d}$, and then changing the signs of $\boldsymbol{b}$ and $\boldsymbol{c}$.

Then $$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

We want to find a matrix A^{-1} so that:

$$A \times A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrix B was too big by a factor of $ad - bc$, as

$$AB = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc)\, I$$

So $$A^{-1} = \frac{1}{ad - bc} B$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To form the inverse we had to divide matrix B by $ad - bc$. As you can never divide by zero, the inverse of a matrix cannot be found if $ad - bc = 0$. Because $ad - bc$ determines whether a matrix has an inverse, it is called the **determinant**. A matrix without an inverse is called **singular** because it has no partner to multiply and make the identity matrix.

Determinant of a 2 × 2 matrix

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** of A is given by

$$\textbf{det}(A) \text{ or} |A| = ad - bc$$

- If $ad - bc = 0$, matrix A has no inverse and is called **singular**.

Inverse of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **inverse of** A is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

or $$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of matrix A:

1 Find $\det(A) = ad - bc$.

2 If $\det(A) = 0$, no inverse exists.

3 Swap the numbers a and d.

4 Change the signs of numbers b and c.

5 Divide each number in the new matrix by the value of $\det(A)$.

Finding the determinant and inverse

Example 10 **Finding the determinant and inverse**

For each matrix below:

- find its determinant
- decide whether the matrix has an inverse
- find the inverse if it exists.

a $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$ **b** $B = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix}$

Solution

a

1 Write the given matrix. $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$

2 Identify a, b, c and d from $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $a = 8, b = 4, c = 3, d = 2$

3 Use $\det(A) = ad - bc$.

4 Substitute in the values of a, b, c and d.

5 Evaluate $\det(A)$.

$$\det(A) = ad - bc = 8 \times 2 - 4 \times 3 = 16 - 12 = 4$$

6 For an inverse to exist the determinant must not equal zero.

An inverse exists.

7 Use $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Substitute in the values of a, b, c and d.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -4 \\ -3 & 8 \end{bmatrix}$$

8 Evaluate each element in A^{-1}.

$$= \begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$$

9 Check your answer by finding $A \times A^{-1}$.

$$A \times A^{-1} = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b

1 Write the given matrix.

$$B = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix}$$

2 Identify a, b, c and d from $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$a = 9$, $b = 6$, $c = 3$, $d = 2$

3 Use $\det(B) = ad - bc$.

4 Substitute in the values of a, b, c and d.

5 Evaluate $\det(B)$.

$$\det(B) = ad - bc = 9 \times 2 - 6 \times 3 = 18 - 18 = 0$$

6 Is the determinant equal to zero? If so, no inverse exists.

No inverse exists.

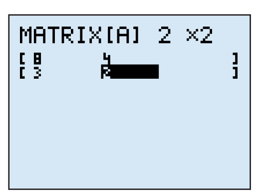
How to find the determinant and the inverse using a graphics calculator

Find the determinant and the inverse of this matrix.

$$A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$$

Steps

1 Enter matrix A.
(If you need help, see 'How to enter a matrix into your graphics calculator' (page 422).)

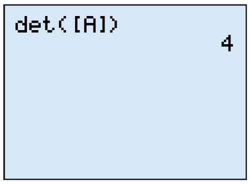


2 Press [2nd] [QUIT] to return to the home screen.

Finding the determinant

1 Press [2nd] [MATRIX] [▸] (**MATH** menu) [1] (det).

Press [2nd] [MATRIX] [1] (for matrix A)[)] [ENTER].



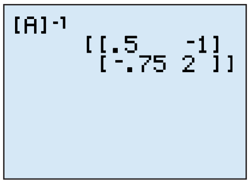
2 Write your answer.

The determinant of A is 4.

Finding the inverse

1 Enter matrix A if you have not already done so.

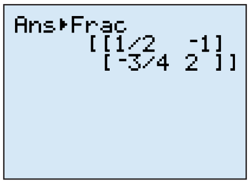
2 Press [2nd] [MATRIX] [1] (for matrix A).
Press [x^{-1}] [ENTER].



3 Write your answer.

The inverse of A is $\begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$.

4 If you want to convert the decimals to fractions, press [MATH] [1] ([▸] Frac) [ENTER].



Exercise 11G

1 Which of these matrices is the identity matrix for matrix multiplication of 2 × 2 matrices?

a $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

2 For each matrix find the following, without using a graphics calculator.

i The determinant of the matrix **ii** The inverse of the matrix, if any

a $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$

e $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ **f** $\begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix}$ **g** $\begin{bmatrix} 10 & -5 \\ 2 & -1 \end{bmatrix}$ **h** $\begin{bmatrix} 8 & -3 \\ 4 & -2 \end{bmatrix}$

3 Use your graphics calculator to find the determinant and inverse of each matrix in Question **2**.

4 Which of the following matrices do *not* have an inverse?

a $\begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **d** $\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$

e $\begin{bmatrix} 3 & -4 \\ 6 & -8 \end{bmatrix}$ **f** $\begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$ **g** $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

5 Use your graphics calculator to find the inverse of each matrix.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 8 \\ 1 & 2 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

11.8 Encoding and decoding information

History has many accounts of the vital role that codes have played in protecting sensitive information used in wars and conspiracies. Mary, Queen of Scots, sent encoded messages from prison to Catholic supporters who planned to overthrow the Protestant Queen Elizabeth. Elizabeth was reluctant to execute her cousin Mary without direct evidence linking her with the plot. The charges were laid by the Principal Secretary, Sir Francis Walsingham. Unfortunately for Mary, Walsingham was also England's spymaster. He used an expert to break the code, and Mary was executed.

In the past, commonly used codes replaced each letter of the alphabet with a randomly chosen number or symbol. The intended recipient could use a list of the changes to change each number back to a letter. The weakness in this type of code is that, in the English language, *E* is the most frequently occurring letter, followed by *T* and then *A*. A table of frequencies for letters can be used to replace numbers occurring with about the same frequency and hence to break the code.

Matrices can be used to encode words so that each letter does *not* have the same number throughout the coded message. This makes the message extremely difficult to decode without knowing the secret encoding matrix. Mary would have survived if she had read the following example. We will use this simple table with the letters numbered in order, but use a matrix to encode the final message.

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	space
15	16	17	18	19	20	21	22	23	24	25	26	27

Example 11 Using matrices to encode and decode messages

Encode the message 'Meet me' then show how to decode the encoded message.
Use the encoding matrix C.

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Solution

Encoding the message

1 Write the letters in a 2 × 4 matrix, M

$$M = \begin{bmatrix} M & E & E & T \\ & M & E & \end{bmatrix}$$

2 Use the table above to replace each letter with its number.

Tip: Do all of the following steps using your graphics calculator.

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

3 Multiply the message matrix M by the secret encoding matrix C.
Notice that the three Es now have three different numbers in this encoded message.
matrix, CM.

$$CM = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

Decoding the message

1 Work out the *inverse* of the secret encoding matrix. Use a calculator.

so the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is $C^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

2 Multiply the encoded message matrix CM by the inverse of the encoding matrix, C^{-1}, to return to the message matrix M.

$$C^{-1} \times CM = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

3 Use the table above to replace each number with its letter.

$$M = \begin{bmatrix} M & E & E & T \\ & M & E & \end{bmatrix}$$

A 2 × 2 matrix can be used to encode any information written as a matrix with two rows. Credit card numbers consisting of 16 digits can be written into a 2 × 8 matrix and encoded using a 2 × 2 matrix.

Tip: Choose a 2 × 2 encoding matrix with small positive numbers so that the numbers in the encoded message do not get too large.

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	space
15	16	17	18	19	20	21	22	23	24	25	26	27

Exercise 11H

1 Use the table for swapping letters with numbers, and matrix C, to *encode* each of the following messages. $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

a $\begin{bmatrix} F & B & I & \\ K & N & O & W \end{bmatrix}$ **b** $\begin{bmatrix} M & A & P & \\ L & O & S & T \end{bmatrix}$ **c** $\begin{bmatrix} A & P & E & \\ F & A & C & E \end{bmatrix}$ **d** $\begin{bmatrix} F & I & N & D \\ & T & O & M \end{bmatrix}$

e $\begin{bmatrix} N & O & & G & U & A & R & D \\ & T & O & N & I & G & H & T \end{bmatrix}$ **f** $\begin{bmatrix} M & E & E & T & & A & N & N \\ A & T & & J & O & H & N & S \end{bmatrix}$

2 Use the letters to numbers table, and the *inverse* of matrix S, to *decode* the following messages. $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

a $\begin{bmatrix} 10 & 27 & 33 & 62 \\ 19 & 45 & 53 & 97 \end{bmatrix}$ **b** $\begin{bmatrix} 28 & 41 & 52 & 59 \\ 47 & 62 & 85 & 91 \end{bmatrix}$ **c** $\begin{bmatrix} 20 & 39 & 63 & 59 \\ 34 & 66 & 101 & 91 \end{bmatrix}$ **d** $\begin{bmatrix} 37 & 65 & 29 & 79 \\ 56 & 109 & 44 & 131 \end{bmatrix}$

e $\begin{bmatrix} 22 & 45 & 48 & 67 & 73 & 15 & 36 & 59 \\ 40 & 75 & 82 & 114 & 119 & 23 & 57 & 91 \end{bmatrix}$ **f** $\begin{bmatrix} 26 & 51 & 30 & 35 & 58 & 23 & 39 & 58 \\ 46 & 84 & 55 & 66 & 89 & 37 & 59 & 89 \end{bmatrix}$

3 Use the encoding matrix B to do the following. $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

a Encode the credit card number written into this 2×8 matrix: $\begin{bmatrix} 3 & 1 & 4 & 7 & 2 & 3 & 8 & 1 \\ 6 & 0 & 5 & 8 & 9 & 3 & 0 & 7 \end{bmatrix}$

b Decode the encoded credit card number received in this matrix: $\begin{bmatrix} 7 & 10 & 8 & 13 & 12 & 0 & 12 & 12 \\ 8 & 19 & 9 & 19 & 20 & 0 & 16 & 21 \end{bmatrix}$

4 Make up an encoded message to fit within a 2×8 matrix. Give a classmate the 2×2 encoding matrix and see whether they can decode it.

11.9 Solving simultaneous equations using matrices

How to solve simultaneous equations using a graphics calculator

Solve to find x and y:

$$5x + 2y = 21$$
$$7x + 3y = 29$$

Steps

1 The two simultaneous equations can be represented by the matrix equation shown.

$$\begin{bmatrix} 5x + 2y \\ 7x + 3y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

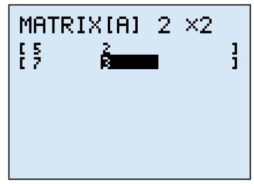
2 The left-hand side of the matrix equation in step 1 can be written as the product of two matrices.

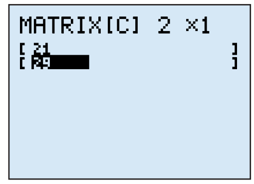
$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

3 Name the matrices as shown.
Matrix B contains the solutions to the simultaneous equations.

$A \times B = C$

4 Enter matrix A and matrix C.
(If you need help, see 'How to enter a matrix into your graphics calculator' (page 422).)



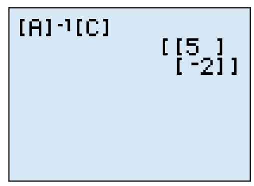


5 We want to find the values of matrix B.

$$\begin{aligned} \text{Since} \quad A \times B &= C \\ A^{-1} \times A \times B &= A^{-1} \times C \\ I \times B &= A^{-1} \times C \\ B &= A^{-1} \times C \end{aligned}$$

$B = A^{-1}C$

6 So press [2nd] [MATRIX] [1] (for A) [x^{-1}] (for its inverse).
Now press [2nd] [MATRIX] [3] (for C) [ENTER].
Note: Order is critical here: $B = A^{-1}C$, *not* CA^{-1}.



7 Write matrix B.

$$B = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

8 Write the solutions to the equations.

So $x = 5$ and $y = -2$.

Exercise 11I

Use matrix methods on your graphics calculator to solve the following simultaneous equations.

1 $3x + 2y = 12$
$5x + y = 13$

2 $4x + 3y = 10$
$x + 2y = 5$

3 $4x - 3y = 10$
$3x + y = 1$

4 $8x + 3y = 50$
$5x + 2y = 32$

5 $6x + 7y = 68$
$4x + 5y = 46$

6 $6x - 5y = -27$
$7x + 4y = -2$

Key ideas and chapter summary

Matrix — A matrix is a rectangular array of numbers set out in rows and columns within square brackets. The **rows** are horizontal; the **columns** are vertical.

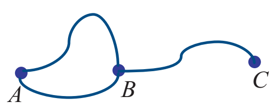
For example, $A = \begin{bmatrix} 5 & 1 & 8 \\ 4 & 7 & 9 \end{bmatrix}$

In the above matrix A the first row is 5 1 8. The second row is 4 7 9.

Order of a matrix — The order (shape) of a matrix is
number of rows × *number of columns*.
The number of rows is always given first.

Elements of a matrix — The elements of a matrix are the numbers within it. The **position of an element** is given by its row and column in the matrix. Element $a_{i,j}$ is in row i and column j. Row is always given first.
For example the element 9 in matrix A (above) is written as $a_{2,3} = 9$.
The 9 is found in the 2nd row and the 3rd column.

Network — A network is a diagram of lines joining points, that shows relationships. A matrix can be used to represent a network and help to solve network problems. The numbers in the matrix show the numbers of connections (lines) between all the points in the network. For example this network and its matrix shows the number of roads connecting towns A, B and C.



$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Equal matrices — Two matrices are equal when they have the same numbers in the same positions. To do this they need to have the same order (shape).

Adding matrices — Matrices of the same order can be added by adding numbers in the same positions.

Subtracting matrices — Matrices of the same order can be subtracted by subtracting numbers in the same positions.

Zero matrix, *O* — A zero matrix is any matrix with zeros in every position.

Scalar multiplication — Scalar multiplication is the multiplication of a matrix by *a number*, which multiplies every number within the matrix.

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Matrix multiplication

For two matrices:

with order: $\underset{m \times n}{A} \quad \underset{p \times q}{B}$

- Matrix multiplication is **defined** (possible) when the inside numbers of the orders are equal; that is, when $n = p$.
- The **order of the answer** is given by the outside numbers of the orders, $m \times q$.

For example:

$$\underset{2 \times 3}{\begin{bmatrix} 5 & 8 & 1 \\ 7 & 6 & 0 \end{bmatrix}} \underset{3 \times 4}{\begin{bmatrix} 9 & 0 & 4 & 2 \\ 2 & 6 & 1 & 5 \\ 7 & 1 & 0 & 6 \end{bmatrix}}$$

Here matrix multiplication is defined because the inside numbers are both 3. The order of the answer matrix will be 2×4.

In general, matrix multiplication is **not commutative**:

$$AB \neq BA$$

Only a **square matrix** (number of rows = number of columns) can be multiplied by itself.

Matrix multiplication methods

The **run and dive** description of matrix multiplication of *AB*:

Add the products of the pairs made as you:

1. **run** along the first row of *A* and **dive** down the first column of *B*
2. repeat running along the first row of *A* and diving down the next column of *B* until all columns of *B* have been used
3. now start running along at the next row of *A* and repeat diving down each column of *B*
4. repeat this routine until all rows of *A* have been used.

It is usually more efficient to do matrix multiplication by using a graphics calculator.

Identity matrix, *I*

An identity matrix *I* behaves like the number 1 in arithmetic. Any matrix multiplied by *I* remains unchanged. For 2×2 matrices,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $AI = A = IA$

Matrix multiplication by the identity matrix is commutative.

Inverse matrix, A^{-1}

When any matrix A is multiplied by its inverse matrix A^{-1}, the answer is I, the identity matrix.

$$A \times A^{-1} = I$$

Determinant, det(A) or $|A|$

The determinant of matrix A is written as det(A) or $|A|$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$.

The determinant helps you to decide (determine) whether the matrix has an inverse. If $\det(A) = 0$, the matrix has no inverse and is called a **singular matrix**.

Inverse of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

or $A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

To find the inverse of matrix A, where $\det(A) \neq 0$:

1 swap the numbers a and d
2 change the signs of numbers b and c
3 divide each number in the new matrix by the value of det(A).

Encoding and decoding

Any information put into a matrix M with 2 rows can be **encoded** by multiplying M by a 2 × 2 encoding matrix C to form matrix CM.
To **decode** the information, multiply matrix CM by the inverse matrix C^{-1}, so that $C^{-1} \times CM$ gives matrix M.

Solving simultaneous equations

Two simultaneous equations, for example:

$$5x + 2y = 21$$
$$7x + 3y = 29,$$

can be written in matrix form as

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$
$$A \times B = C$$

The equations can be solved (for x and y) by finding the values in matrix B, as

$$B = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \times C$$

Order is critical: $B = A^{-1}C$, *not* CA^{-1}.
This is best done using a graphics calculator.

Review

Skills check

Having completed this chapter you should be able to:

- state the order of a given matrix
- describe the location of an element within a matrix
- represent a network as a matrix
- decide whether two matrices are equal
- add and subtract matrices
- do scalar multiplication of a matrix
- identify a zero matrix
- decide whether it is possible to do matrix multiplication with two given matrices
- give the order of the matrix resulting from matrix multiplication
- perform matrix multiplication
- state the identity matrix for a 2 × 2 matrix and know its properties
- decide whether a matrix has an inverse
- find the inverse of a matrix if it exists
- encode and decode information using matrices
- use matrix methods to solve simultaneous equations.

Multiple-choice questions

Use the matrices F, G, H in Questions 1 and 2

$$F = \begin{bmatrix} 4 & 8 & 6 \\ 5 & 1 & 7 \end{bmatrix} \quad G = \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$$

1 The order of matrix F is:

A 6 **B** 2×3 **C** 3×2 **D** $2 + 3$ **E** $3 + 2$

2 The element $f_{2,1}$ is:

A 3 **B** 2 **C** 8 **D** 1 **E** 5

3 Three students were asked the number of electronic devices their family owned. The results are shown in the matrix.

$$\begin{array}{r} \\ Caroline \\ Delia \\ Emir \end{array} \begin{array}{c} \begin{array}{ccc} TVs & VCRs & PCs \end{array} \\ \begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix} \end{array}$$

The number of PCs owned by Emir's family is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Review

4 The matrix gives the numbers of ways of travelling from one town to another. The number of ways of travelling from town B to town C is:

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \end{array}$$

A 0 **B** 1 **C** 2 **D** 3 **E** 4

5 For these two matrices to be equal, the required value of x is:

$$\begin{bmatrix} 4 & 3x \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

A 2 **B** 3 **C** 4 **D** 6 **E** 18

Use matrices M and N in Questions 6 to 10

$$M = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix} \quad N = \begin{bmatrix} 5 & -2 \\ 1 & 0 \end{bmatrix}$$

6 The matrix $M + N$ is:

A $\begin{bmatrix} 12 & 8 \\ 5 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 12 & 4 \\ 4 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} 12 & 4 \\ 4 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 12 & 4 \\ 5 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 12 & 4 \\ 5 & 0 \end{bmatrix}$

7 The matrix $M - N$ is:

A $\begin{bmatrix} 2 & 8 \\ 3 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 8 \\ 3 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 8 \\ 4 & 3 \end{bmatrix}$

8 The matrix $N - N$ is:

A 0 **B** [0] **C** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & -2 \\ 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} -5 & 2 \\ -1 & 0 \end{bmatrix}$

9 The matrix $2N$ is:

A $\begin{bmatrix} 10 & -4 \\ 2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 7 & 0 \\ 3 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 10 & -4 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 10 & -2 \\ 2 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 10 & -4 \\ 2 & 2 \end{bmatrix}$

10 The matrix $2M + N$ is:

A $\begin{bmatrix} 14 & 10 \\ 7 & 5 \end{bmatrix}$ **B** $\begin{bmatrix} 14 & 6 \\ 7 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 24 & 8 \\ 10 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 19 & 14 \\ 9 & 6 \end{bmatrix}$ **E** $\begin{bmatrix} 19 & 10 \\ 9 & 6 \end{bmatrix}$

Use the matrices P, Q, R, S in Questions 11 to 14

$$P = \begin{bmatrix} 5 & 4 & 1 \\ 7 & 6 & 8 \end{bmatrix} \quad Q = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \quad R = \begin{bmatrix} 4 & 7 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

11 Matrix multiplication is *not* defined for:

A PQ **B** SS **C** SP **D** PS **E** RS

12 The order of matrix QR is:

A 1×1 **B** 3×2 **C** 2×3 **D** 6 **E** 5

Review

13 Which of the following matrix multiplications gives a matrix of order 1×3?

A QQ **B** RQ **C** PR **D** QR **E** RP

14 The matrix multiplication PQ gives the matrix:

A $\begin{bmatrix} 34 \\ 50 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 24 & 0 \\ 14 & 36 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 10 & 14 \\ 24 & 0 \\ 36 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 4 & 1 \\ 7 & 6 & 8 \\ 2 & 6 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 34 & 50 \end{bmatrix}$

15 The identity matrix for 2×2 matrices is:

A $[1]$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 The inverse of matrix $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ is:

A $\begin{bmatrix} -4 & -5 \\ -2 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **C** $\frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ **E** $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

Short-answer questions

Use matrix A in Questions 1 to 4

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 3 & 4 & 7 & 9 \end{bmatrix}$$

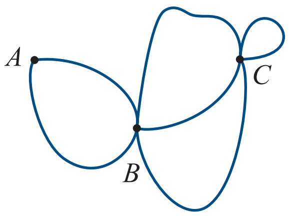
1 State the order of matrix A.

2 Identify the element $a_{1,3}$.

3 If $C = [5 \quad 6]$, find CA.

4 If the order of a matrix B was 4×1, what would be the order of the matrix resulting from AB?

5 Roads are shown joining towns A, B and C. Use a matrix to record the numbers of ways to travel from town to town. (The loop road returning to town C means that there are two ways of travelling from C to C.)



6 The two matrices shown are equal. Find the values of w, x, y and z.

$$\begin{bmatrix} x & 3y \\ 9 & w-2 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ z+1 & 0 \end{bmatrix}$$

Review

7 Use the matrices below.

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 5 \\ 7 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find:

a $3A$ **b** $A + B$ **c** $B - A$ **d** $2A + B$ **e** $A - A$ **f** AB
g BA **h** A^{-1} **i** A^2 **j** AI **k** $\det(A)$ **l** AA^{-1}.

Extended-response questions

1 Farms A and B have their livestock numbers recorded in the matrix shown.

$$\begin{array}{c} \\ \textit{Farm A} \\ \textit{Farm B} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Cattle} & \textit{Pigs} & \textit{Sheep} \end{array} \\ \begin{bmatrix} 420 & 50 & 100 \\ 300 & 40 & 220 \end{bmatrix} \end{array}$$

a How many pigs are on Farm B?
b What is the total number of sheep on both farms?
c Which farm has the largest total number of livestock?

2 A bakery recorded the sales for Shop A and Shop B of cakes, pies and rolls in a Sales matrix, S. The prices were recorded in the Prices matrix, P.

$$\text{Sales, } S = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Cakes} & \textit{Pies} & \textit{Rolls} \end{array} \\ \begin{bmatrix} 12 & 25 & 18 \\ 15 & 21 & 16 \end{bmatrix} \end{array} \qquad \text{Prices, } P = \begin{array}{c} \textit{Cakes} \\ \textit{Pies} \\ \textit{Rolls} \end{array} \overset{\$}{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

a How many pies were sold by Shop B?
b What is the selling price of pies?
c Calculate the matrix product SP.
d What information is contained in matrix SP?
e Which shop had the largest income from its sales? How much were its takings?

3 Patsy and Geoff decided to participate in a charity fun run.

a Patsy plans to walk for 4 hours and jog for 1 hour. Geoff plans to walk for 3 hours and jog for 2 hours. Write out matrix A, filling in the missing information.

$$A = \begin{array}{c} \\ \textit{Patsy} \\ \textit{Geoff} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Hours walking} & \textit{Hours jogging} \end{array} \\ \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array}$$

b Walking raises \$2 per hour and consumes 1500 kJ/h (kilojoules per hour). Jogging raises \$3 per hour and consumes 2500 kJ/h. Write out matrix B, filling in the missing information.

$$B = \begin{array}{c} \\ \textit{Walking} \\ \textit{Jogging} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \textit{kJ} \end{array} \\ \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array}$$

c Use matrix multiplication to find a matrix that shows the money raised and the kilojoules consumed by each person.

4 **a** The 16 digits of a credit card are recorded in matrix D.

$$D = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 2 & 1 & 1 \\ 2 & 1 & 4 & 2 & 2 & 1 & 0 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Use matrix A to encode the credit card numbers. Give the matrix with the encoded credit card numbers.

b Matrix E consists of 16 credit card numbers that have been encoded using matrix A. Decode matrix E to the matrix with the original credit card numbers.

$$E = \begin{bmatrix} 5 & 6 & 3 & 9 & 2 & 7 & 4 & 4 \\ 3 & 3 & 2 & 5 & 2 & 5 & 2 & 3 \end{bmatrix}$$

5 We are told that 2 apples and 3 bananas cost \$6. This can be represented by the equation

$$2x + 3y = 6$$

where x represents the cost an apple and y the cost of a banana.

a Write an equation for: 6 apples and 5 bananas cost \$14.

b With your equation and the equation given, use matrix methods on your graphics calculator to find the cost of an apple and the cost of a banana.